

A SPHERE HARD TO CUT

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ABSTRACT. We show that for any $\epsilon, M > 0$ there is a Riemannian 3-sphere S of volume 1, such that any (not necessarily connected) surface separating S in two regions of volume greater than ϵ , has area greater than M .

1. INTRODUCTION

Glynn-Adey and Zhu show in [4] that for any $\epsilon > 0, M > 0$ there is a Riemannian 3-ball B of volume 1 such that any smooth disk separating B in two regions of volume greater than ϵ has area greater than M . We prove the same result here both for the 3-ball and the 3-sphere for separations by arbitrary surfaces and not just disks. Glynn-Adey and Zhu assume further that the ball B has bounded diameter and boundary surface area but these are properties that are easy to arrange in general modifying slightly the ball B .

These results contrast with the situation in dimension 2. Liokumovich, Nabutovsky and Rotman showed in [7] that if D is a Riemannian 2-disc there is a simple arc of length bounded by $2\sqrt{3}\sqrt{\text{area}(D)} + \delta$ which cuts the disc into two regions of area greater than $\text{area}(D)/4 - \delta$ where δ is any positive real. A similar result was shown in [9] for the sphere. The results in [7] were prompted by a question of Gromov [5] and Frankel-Katz [2] concerning bounding the length of contracting homotopies of a 2-disk.

Balacheff-Sabourau [1] showed that there is some $c > 0$ such that any Riemannian surface M of genus g can be separated in two domains of equal area by a 1-cycle of length bounded by $c\sqrt{g+1}\sqrt{\text{area}(M)}$. Liokumovich [6] on the other hand showed that given $C > 0$ and a closed surface M there is a Riemannian metric of diameter 1 on M such that any 1-cycle splitting it into two regions of equal area has length greater than C .

2. A SPHERE HARD TO CUT

Definition . Let B be a Riemannian 3-ball. If $F \subset B$ is a smoothly embedded orientable surface with boundary we say that F *separates* B if $F \cap \partial B = \partial F$.

If F is a surface separating a Riemannian 3-ball B we say that F *cuts an ϵ -piece* of B if $B - F$ can be written as a union of two disjoint open sets U, V both of which have volume greater than ϵ .

We define similarly what it means for a closed surface to cut an ϵ -piece of a Riemannian 3-sphere.

Our construction relies on the existence of expander graphs. We recall now the definition of expanders. Let $\Gamma = (V, E)$ be a graph. For $S, T \subseteq V$ denote the set of all edges between S and T by

$$E(S, T) = \{(u, v) : u \in S, v \in T, (u, v) \in E\}.$$

Definition . The *edge boundary* of a set $S \subseteq V$, denoted ∂S is defined as $\partial S = E(S, S^c)$.

A k -regular graph $\Gamma = (V, E)$ is called a *c -expander graph* if for all $S \subset V$ with $|S| \leq |V|/2$, $|\partial S| \geq c|S|$.

Pinsker [8] has shown that there is a $c > 0$ such that for any n large enough there is a 3-regular expander graph with n -vertices.

Consider a 3-regular c -expander graph Γ_n with n^3 vertices. We give a way to ‘thicken’ this graph, i.e. replace it by a Riemannian handlebody. For each vertex we pick a Euclidean 3-ball B_v of radius $1/n$. Recall that the volume of this ball is $4\pi/3 \cdot (1/n)^3$. Let S_v be the boundary sphere of B_v . If l is an equator of S_v we pick 3 equidistant points e_1, e_2, e_3 on l and we consider 3 disjoint (spherical) discs on S_v with centers e_1, e_2, e_3 and radii equal to $1/n$. Clearly these discs are disjoint. Now to each edge E_i in Γ leaving v we associate the disc with center e_i . If an edge e joins the vertices v, w of Γ we identify the discs of the balls B_v, B_w corresponding to this edge.

In this way we obtain a handlebody Σ_n . Note that $\partial \Sigma_n \cap B_v$ is a pair of pants. We will refer to B_v later on as a filled in pair of pants and we will call the discs with centers e_1, e_2, e_3 on S_v the holes of this pair of pants. We note that the area of S_v is $4\pi(1/n)^2$ and the area of S_v minus the 3 spherical discs is

$$4\pi(1/n)^2 - 6\pi(1/n)^2(1 - \sin 0.5) = \pi(1/n)^2(6 \sin 0.5 - 2).$$

By changing the metric of Σ_n slightly we get a smooth handlebody, denoted still by Σ_n , of volume $4\pi/3$. Finally by gluing appropriately thickened discs to this handlebody we obtain a ball B_n . We may assume that this gluing operation changes the volume of B_n and the area of

its boundary by a negligible amount. We may pick a simple curve γ on ∂B_n such that every point of ∂B_n is at distance at most $1/n$ from γ . By gluing a thickened disk of diameter $1/n$ and negligible volume to ∂B_n along γ we obtain a new ball of arbitrarily small diameter. We still denote this 3-ball by B_n . In fact it follows also directly by the properties of expander graphs that the diameter of B_n is bounded.

We double B_n along its boundary to obtain a 3-sphere. By changing the metric slightly along the doubling locus we may ensure that we obtain a smooth sphere S_n of volume $8\pi/3$.

Theorem 2.1. *Given $\epsilon, M > 0$ there is some n such that any surface that cuts an ϵ -piece of B_n (or S_n) has area greater than M .*

Proof. We may (and will) assume that $\epsilon < 1/100$. Let F be a (not necessarily connected) surface cutting an ϵ -piece of B_n . So $B_n - F = U_1 \cup U_2$ with U_1, U_2 open of volume greater than ϵ . We denote by Q_1, Q_2 the closures of U_1, U_2 respectively. Without loss of generality we assume that $\text{vol}(U_2) \geq \text{vol}(U_1)$.

We note that B_n contains a handlebody Σ_n which is a union of filled in pairs of pants B_v —one for each vertex of the graph Γ_n . Clearly $S_v \cap \partial \Sigma_n$ is a pair of pants with 3 holes.

Let B_v be one such (filled in) pair of pants. Its volume is $4\pi/3n^3$. By the solution of the isoperimetric problem for a ball ([10]) if a surface cuts an $\epsilon 4\pi/3n^3$ piece of B_v then its area is greater than $(4\pi\epsilon/3n^3)^{2/3} \geq \epsilon/n^2$.

Let's say that for n_1 filled in pairs of pants F cuts an ϵ/n^3 piece and that for n_2 filled in pairs of pants more than $\frac{4\pi(1-\epsilon)}{3n^3}$ of their volume is contained in U_1 . Since $\text{vol}(U_1) \leq \text{vol}(U_2)$

$$n_2 \leq 2\epsilon n^3 \leq n^3/2$$

We distinguish two cases.

Case 1. $n_1 \geq \epsilon n^3/2$. Since the area of intersection of F with each one of these n_1 filled in pairs of pants is greater than ϵ/n^2 the area of F is greater than $\epsilon^2 n/2$ which clearly tends to infinity as $n \rightarrow \infty$.

Case 2. $n_1 < \epsilon n^3/2$. Since $\text{vol}(U_1) > \epsilon$ we have that $n_2 \geq \epsilon n^3/2$. Let's denote this set of n_2 -filled pairs of pants by A . Let B_v be in A , and let $U_v = B_v \cap U_1$. Since

$$\text{vol}(U_v) \geq \frac{4\pi(1-\epsilon)}{3n^3}$$

by the Euclidean isoperimetric inequality the boundary of U_v has area at least

$$\frac{4\pi(1-\epsilon)^{2/3}}{n^2}.$$

Since $\epsilon < 1/100$ it follows that if the area of $F \cap B_v$ is less than $\epsilon/2n^2$ then U_1 intersects non-trivially all 3 holes of the filled-in pair of pants.

In fact since the area of a spherical cap is given by $2\pi rh$ where r is the radius and h the height, the area of the intersection of U_1 with a hole is greater than

$$\frac{2\pi}{4n^2} > \frac{1}{n^2} \quad (*).$$

Let's denote by A_1 the set of filled-in pair of pants in A for which the area of intersection of $F \cap B_v$ is more than $\epsilon/2n^2$ and let $A_2 = A - A_1$. We set $k_1 = |A_1|$, $k_2 = |A_2|$ and note that

$$k_1 + k_2 = n_2 \geq \frac{\epsilon n^3}{2}.$$

If $k_1 \geq \epsilon n^3/4$ then we see that the area of F is greater than $\epsilon^2 n/8$ which clearly tends to infinity as $n \rightarrow \infty$. Otherwise $k_2 \geq \epsilon n^3/4$. By the expander property (and since $k_2 \leq n^3/2$) the (not necessarily connected) union of filled in pairs of pants in A_2 , Σ , has a boundary that consists of at least

$$ck_2 \geq \frac{c\epsilon n^3}{4}$$

holes. Let B_v be a filled-in pair of pants adjacent to one of these holes. Clearly B_v intersects U_1 . We claim that the area of $B_v \cap F$ is at least ϵ/n^2 . This is clear if F cuts an $4\pi\epsilon/3n^3$ piece of B_v or if B_v lies in A_1 . If this is not the case then more than $(1-\epsilon)4\pi/n^3$ of the volume of B_v is contained in U_2 . Let O_v be the center of B_v . Let's denote by C_r the sphere with radius r and center O_v . Let l_r be the length of the intersection of F with C_r . If $l_r > 1/10n$ for all r with $1/n > r > 9/10n$ then by the co-area formula ([3], 3.2.22) the area of $F \cap B_v$ is greater than $1/100n^2 > \epsilon/n^2$. Otherwise we consider an $r_0 \in (9/10n, 1/n)$ for which l_{r_0} is smaller than $1/10n$. We consider the portion F_1 of F between C_{r_0} and the boundary of B_v and we fill the holes of F_1 lying on C_{r_0} by minimal area discs. The total area of these disks is smaller than $\frac{\pi}{100n^2}$. Let's call the surface obtained this way by F_2 . Note that F_2 separates $U_1 \cap B_v$ from O_v . Let f be the radial projection from O_v to $C_1 = S_v$. Clearly $f(F_2)$ contains $S_v \cap U_1$ and by inequality (*) the area of $S_v \cap U_1$ is greater than $\frac{1}{n^2}$. Also f is Lipschitz with Lipschitz

constant less than 2. So the area of $f(F_2 - F_1)$ is less than $\frac{\pi}{50n^2}$. It follows that the area of F_1 is greater than

$$\frac{1}{4n^2}$$

so the area of $F \cap B_v$ is greater than ϵ/n^2 in this case too.

It follows as before that the area of F is at least

$$\frac{c\epsilon n^3}{4} \cdot \frac{\epsilon}{n^2} = \frac{c\epsilon^2 n}{4}$$

which clearly tends to infinity as $n \rightarrow \infty$.

The result for the 3-sphere S_n follows immediately from B_n as S_n is a union of two copies of B_n and if a surface cuts an ϵ -piece of S_n it cuts an $\epsilon/2$ piece in one of these two copies of B_n . Finally clearly we may normalize the volume of S_n, B_n to 1.

□

Remark 1. In [4] it is assumed additionally that the surface area and the diameter of the ball B_n is bounded. However both these properties are easy to arrange. As for the surface area one may excise a small ball from the 3-sphere S_n in the proof above and obtain a ball B such that the area of ∂B is arbitrarily small. By construction B_n, S_n have diameter less than 1. In fact given any ball (in any dimension ≥ 3) one can easily decrease its diameter by surgery: one may cut out a thickened simple curve and glue back in a ball with small diameter. This has no effect on the volume- or separation properties of the ball. Even though we stated our result only for dimension 3 the same construction applies for spheres (balls) of any dimension $n \geq 3$.

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